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LETTER TO THE EDITOR

Retarded and apparent positions: their geometry

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Abstract. Light from a fast moving point object does not arrive at an observer's eye from the direction where the object currently is, but from the direction of its retarded position. If stereoscopic vision is used however, so that distance can be judged, the object does not generally appear to be at its retarded position, but at a different distance away, *depending on the orientation of the 'eye pair'*. These effects are analysed geometrically, invoking elementary plane projective geometry.

Due to the finite speed of light c, an observer's eye E does not see a moving point object O at its current position, but in the direction of the *retarded* position R where the object was when the light, currently arriving at the eye, departed from it. If the object is in uniform motion at speed v along a straight line the retarded position is easy to find analytically—nothing more than the cosine rule for triangles is required. It is also easy to construct R geometrically by drawing a circle of radius vy/c (figure 1) about O. The tangent line ET to it then intersects the path of the object at the point R. (The other tangent would construct the advanced position.)

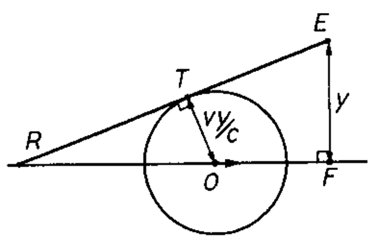


Figure 1. Circle tangent construction of the retarded position R. The radius of the circle is derived by similar triangles: $v/c = RO/RE = OT/EF$.

This geometrical construction acquires extra significance in the case of vision with two eyes (stereoscopic or binocular vision) allowing distance to be judged. The fact that the retarded position is generally slightly different for each eye (at $R \pm dR/2$, say), means the object does not appear to lie at R, but at a completely different distance along the line of sight ER (even, possibly, beyond E or R). (This effect bears a superficial resemblance to the familiar one of apparent depth when viewing a (stationary) submerged object, or indeed when using any optical instrument such as a magnifying lens.) For example, suppose the eyes have a fixed *infinitesimal* separation vector parallel to the path of the object. They then see the passing object not at R, but at the *apparent* position T. This is obvious from the construction: the rays reaching each eye are both tangent to the circle at nearly the same place—in the limit, at T itself. They therefore intersect each other there, and the object appears to be there (figure 2). As the object moves past, T slides around half the circle centred on the object; the (half) circle is the envelope, or caustic, of the family of successive rays.

In contrast to the apparent depth effect for a submerged object, the apparent position of the moving object *changes with the orientation of the pair of eyes* (i.e. rotation of their separation vector). If the eye pair rotates in the plane so as to face the apparent object (i.e. so the normal to their separation vector lies along ER) then the retarded position is the same for both eyes (they lie at the same distance from it) and the apparent position does lie at R. (Two different circles would be involved in the tangent construction here since the eyes have different y values). This case was noted by Terrell [1] in the well known paper in which he discovered the conformality of the relativistic transformation of apparent directions between the reference frames of E and O (as did Penrose [2], independently). The transformation is not involved here since we work in the E frame only.

Another special apparent position is realized (in principle) if the eye pair is rotated to be along the line ER. Then the apparent position is at zero distance, actually at the eyes themselves. This follows because the ray through one eye (the more distant one, say) passes through the nearer eye, and therefore intersects the ray through the nearer one there. (Again here, two different circles would be required to deduce this apparent position from the tangent construction.)

In what follows we shall state, and then derive the apparent position A for general orientation of the eye pair coplanar with the object's path. (For non-coplanarity there is usually no apparent position because the rays reaching the two eyes are skew—they do not intersect.) The result is most naturally expressed geometrically (figure 3). A fixed 'projection point' P is constructed as the intersection of the line TF with the normal to RE through E. The apparent position A is the projection from P, onto the line of sight RE, of the intersection point N of the normal to the eyes with the object's path.

As the eye pair is rotated through 180°, N moves all along the object's path, and hence A moves all along the line of sight ER (out to $\pm$ infinity). For instance, for the three cases above: (i) when N is at F, A is at T, (ii) when N is at R, A is at R, (iii) when N is at Q, A is at E. A less obvious special case is that where the eyes are parallel to OE; then A can be shown to lie at infinity (positive or negative infinity being equivalent).

A derivation runs as follows. Denote the displacement OE by $\boldsymbol{r}=(x, y)$, OA by $\boldsymbol{r}'$, and the eye pair normal by a unit vector $\boldsymbol{n}$ (its sign is immaterial). We seek $\boldsymbol{r}'(\boldsymbol{r}, \boldsymbol{n})$. Define the function $\tau(\boldsymbol{r})$ as the time of flight of the light reaching the eyes at $\boldsymbol{r}$, or retardation time, that is, the distance RE divided by c. We do not need to use the

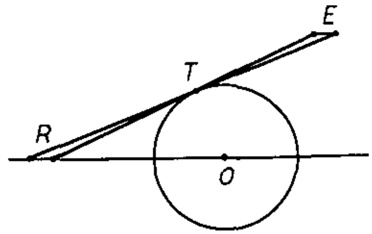
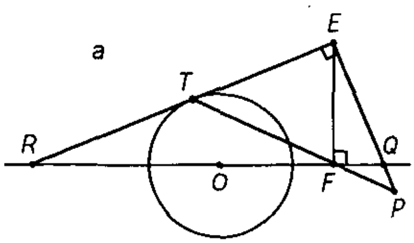


Figure 2. Apparent position at T for eye separation parallel to the object's direction of motion.



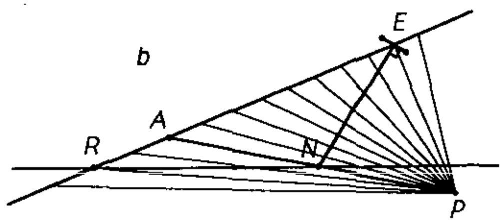


Figure 3. (*a*) Construction of the projection point *P*. (*b*) The apparent position *A* is the projection of the eye normal intercept *N*.

explicit form of this function,

$$\tau(x, y) = (\gamma^2/c)[\sqrt{x^2+y^2/\gamma^2} + xv/c] \qquad (\gamma = 1/\sqrt{1-v^2/c^2}) \tag{1}$$

but note that its contours are non-concentric circles (the 'wavefronts' of the moving source) centred on the retarded position $R(\boldsymbol{r})$ (figure 4). The ray reaching an eye at $\boldsymbol{r}$ lies along the radius of the circle through $\boldsymbol{r}$, that is, it is along $\nabla\tau(\boldsymbol{r})$. The apparent position $\boldsymbol{r}'$ lies, by definition, at the intersection of the rays reaching the two infinitesimally separated eyes, separation $d\boldsymbol{r}$. Thus $\boldsymbol{r}'$ obeys the two equations

$$(\boldsymbol{r}'-\boldsymbol{r}) \wedge \nabla\tau(\boldsymbol{r}) = 0 \tag{2}$$

$$(\boldsymbol{r}'-\boldsymbol{r}-d\boldsymbol{r}) \wedge \nabla\tau(\boldsymbol{r}+d\boldsymbol{r}) = 0. \tag{3}$$

Subtraction of these yields $d\boldsymbol{r}$ dotted with the derivative of (2), so that, in index notation

$$dr_i\, \varepsilon_{ij} \nabla_j \tau = dr_i (r'_j - r_j) \varepsilon_{jk} \nabla_k \nabla_i \tau. \tag{4}$$

Replacing dr_i by $\varepsilon_{ij} n_j$, and setting $(\boldsymbol{r}'-\boldsymbol{r}) = \Lambda\nabla\tau$ from (2), we have,

$$\Lambda = -n_i \nabla_i \tau / [(\varepsilon_{jk} n_k)(\nabla_j \nabla_l \tau)(\varepsilon_{lm} \nabla_m \tau)] \tag{5}$$

This form $\boldsymbol{n}\cdot\boldsymbol{a}/\boldsymbol{n}\cdot\boldsymbol{b}$ can be recognized from projective geometry [3, 4], whose required essentials we first summarize in the next paragraph. Actually, for us (with τ given by (1)), $\boldsymbol{b}$ has a simple direction, though this is not needed; because the second

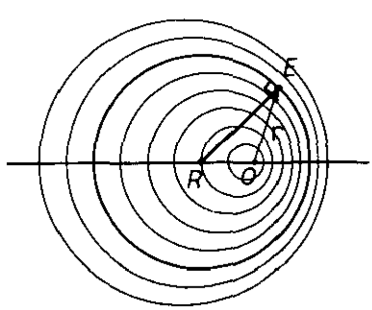


Figure 4. Contours of the retardation time $\tau(\boldsymbol{r})$. The normal to the contour at $\boldsymbol{r}$, i.e. the circle radius, is the ray reaching an eye at $\boldsymbol{r}$.

derivative matrix in the denominator of (5) has a zero eigenvalue in the direction of $\boldsymbol{r}$, $\boldsymbol{b}$ lies parallel to $\boldsymbol{r}$.

Plane projective geometry, studies, in two dimensions, the mapping from points of one straight line to points of another by projection from a point source (a 'perspectivity'). The mapping produced by any sequence of perspectivities (a 'projectivity') can actually be reproduced, almost always, by just *two* appropriately chosen perspectivities (involving, therefore, just two sources and a single intermediate line). If, moreover, the desired resultant mapping has the simple feature that the intersection point of initial and final line is mapped to itself, then a single perspectivity will achieve the projectivity. A projectivity is fully specified by the images of any three points. Algebraically, the mapping takes the form $\Lambda = (a\lambda + b)/(c\lambda + d)$, where λ and Λ measure distance from arbitrary origins on the initial and final lines.

For us, the final line is the line of sight ER, and the initial line can be taken as the object's path. If then, $\boldsymbol{n}$ is specified by its intersection with the object's path, at coordinate λ (position N), its components can be written in terms of λ, and (5) takes the required form. We already know the images of three points: $F \to T$, $R \to R$, $Q \to E$, and since the intersection point R is indeed mapped to itself, only a single projection is required, with the projection point P determined by the F and Q mappings. The simplicity derives from the special feature mentioned after equation (5).

We conclude with miscellaneous remarks.

(i) The function $\boldsymbol{r}'(\boldsymbol{r}, \boldsymbol{n})$ satisfies the scaling $\boldsymbol{r}'(\mu \boldsymbol{r}, \boldsymbol{n}) = \mu \boldsymbol{r}'(\boldsymbol{r}, \boldsymbol{n})$, for $\mu > 0$. This follows from the fact that $\tau(\mu \boldsymbol{r}) = \mu\tau(\boldsymbol{r})$, from (1), so that from (5), $(\boldsymbol{r}' - \boldsymbol{r}) = \Lambda \nabla \tau$ scales likewise. Thus for fixed $\boldsymbol{n}$, radial lines from O map to radial lines again, rotating through some angle, and having a change of scale.

(ii) Although the presentation has been couched in terms of light, no relativity has been required and the analysis applies equally well to a moving source of any non-dispersive wave, sound for instance. In this case supersonic motion of the source can be considered. Inside the Mach cone (of half angle $\sin^{-1}(c/v)$) the function $\tau(\boldsymbol{r})$ is double valued since there are two separate retarded positions along the source's path. (For $\boldsymbol{r}$ outside the cone there are none). The two are generated via the two tangents to the circle of radius $vy/c(>y)$ described in the first paragraph and, proceeding just as before for each, two apparent positions can be found. In practice through, for sound, locating the source would probably be done by timing, using several 'ears' rather than by simultaneous direction-finding eyes. In that case the idea of apparent position would not arise.

(iii) For a non-uniformly moving object the algebra of the apparent position (equations (2) to (5)) remains unchanged, with the appropriate function $\tau(\boldsymbol{r})$ inserted in place of (1). The geometric construction of figure 3 also applies but with a fake position O, no longer the present position of the object, but its 'uniformly extrapolated' position; where the object would be if it had maintained its velocity at the retarded time for the duration τ. Of course the geometric construction of figure 1 is no use for finding R, but once R, τ, and the velocity at the retarded time are supplied, T is constructed by dropping a perpendicular from (the fake) O onto RE, and the geometry of figure 3 proceeds. The justification for this geometric generalization is that two consecutive circular contours of τ suffice for finding A. (Algebraically, the $\nabla\tau$ in (2) can be multiplied by any non-zero function, for example it can be rendered a unit vector.)

(iv) For small velocities of motion the retarded position lies close to the present position, and the construction circle of figure 1 is small. This means that the projection point P is close to the object's path, so that for all eye pair orientations except those with N close to Q, the apparent position A is close to R and O, as expected.

(v) Perhaps less artificial, practically, than the 'pair of eyes' observing the fast moving object would be a recorded ***hologram***. This, acting like a continuum of eyes, is subject to the same deception.

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[2] Penrose R 1959 *Proc. Camb. Phil. Soc.* **55** 137–9
[3] Coxeter H S M 1961 *Geometry* (New York: Wiley)
[4] Coxeter H S M 1964 *Projective Geometry* (New York: Blaisedell)